An Unexpected Electrovac Solution with the Negative Cosmological Constant

Jerzy Klemens Kowalczyński

Institute of Physics, Polish Academy of Sciences,

Al. Lotników 32/46, 02-668 Warsaw, Poland

E-mail address: jkowal@ifpan.edu.pl

Abstract. An exact solution of the current-free Einstein–Maxwell equations with the cosmological constant is presented. The solution is of Petrov type D, includes the negative cosmological constant, and could be a "background addition" to the present-day models of the universe. It has a surprising property such that its electromagnetic field and cosmological constant are interdependent (this constant is proportional to the energy density of this field), which may suggest a new way of measuring the constant in question. The solution describes a constant electromagnetic background with a preferred direction in the universe, and defines the entire lifetime of the universe as a simple function of the negative cosmological constant. According to our solution the absolute value of this constant should be considerably lower than that recently estimated, when astrophysical data are taken into account. Our solution is a special case of that published by Bertotti in 1959. His solution (in terms of which the cosmological constant and the background electromagnetic field are independent) and its two other special cases, i.e. the conformally flat Robinson solution (1959) and the one which is the counterpart of our solution with the positive cosmological constant, are briefly discussed.

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The present-day descriptions of the universe are an important domain of general relativity. Apart from several well-known homogeneous models of the universe, there is a large number of inhomogeneous ones [1]. Both classes are considered with or without the cosmological constant, Λ, which has advocates and opponents. Some advocates even consider this constant to be inevitable in the models [2]. The question of Λ in cosmology is presented in detail by Carroll, Press, and Turner in their review article [2] provided with a huge list of topical literature. The effects of the cosmological constant on the homogeneous models are also described in Ref. [3] with lucid and concise summaries on pp. 746, 747, 773, and 774. Examples of the strong effect in the case of inhomogeneous models are presented on pp. 25 and 27 in Ref. [1]. In general, the models with and without Λ may considerably differ.

The problem of whether the cosmological constant exists is therefore one of the central problems of cosmology today. In order to settle it, astrophysicists implant their observational data (mean mass-energy density of the universe, deceleration parameter, Hubble "constant") in equations resulting from the so-called cosmological solutions of the Einstein equations for dust or fluid. However, there is considerable discrepancy between the observational data (see, e.g., Refs. [3,4] and p. 587 in Ref. [5]). Besides, there is no consensus as to which one of those solutions (i.e. cosmological models) should be taken into account as the best approximation to reality. The possible values of Λ, positive or negative, are, therefore, only roughly estimated. The recent estimation [6], based on Refs. [2,4], is

$$0 \leq |\Lambda| \lesssim 2.2 \times 10^{-56}\ \mathrm{cm}^{-2} \tag{1}$$

in the case of $\Lambda \leq 0$, which is the one we are interested in here. (For $\Lambda \geq 0$ the upper limit is two times higher [4,6]). There is no lower limit different from zero, i.e. the models with $\Lambda = 0$ are not excluded.

In consequence, today the estimation of Λ is not only complicated but also

uncertain and indirect. It appears, however, that the Einstein–Maxwell theory points to a different approach, possibly simpler and more direct, since it consists in measuring a constant electromagnetic background of the universe. Such a possibility is illustrated below for $\Lambda \leq 0$, i.e. for the case when the presence of Λ decelerates the expansion of the universe.

The following simple metric form

$$ds^2 = dx^2 + dy^2 + 2\left(1 + \Lambda uv\right)^{-2} du\, dv \tag{2}$$

and electromagnetic field tensor

$$F_{xy} = p, \quad F_{uv} = \left(1 + \Lambda uv\right)^{-2} q, \quad F_{xu} = F_{xv} = F_{yu} = F_{yv} = 0, \tag{3}$$

with

$$\Lambda = -c^{-4} G \left(p^2 + q^2\right), \tag{4}$$

where p and q are real constants, c is the speed of light in vacuum, and G is the Newtonian gravitational constant, are an exact solution of the current-free Einstein–Maxwell equations with the cosmological constant

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = g_{\mu\nu}\Lambda + 2c^{-4} G \left(F_{\sigma\mu} F_{\nu}{}^{\sigma} + \tfrac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}\right),$$

$$F_{[\mu\nu,\sigma]} = 0, \qquad F^{\mu\nu}{}_{;\nu} = 0, \tag{5}$$

and where the signature $+++-$ and convention $R_{\mu\nu} := R^{\sigma}{}_{\mu\nu\sigma}$ are assumed. The invariant and pseudoinvariant of the electromagnetic field are

$$F_{\mu\nu} F^{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right) = 2\left(p^2 - q^2\right), \quad F_{\mu\nu} \widetilde{F}^{\mu\nu} = 4\mathbf{E}\mathbf{B} = 4pq, \tag{6}$$

where $\widetilde{F}^{\mu\nu}$ is the dual of $F_{\mu\nu}$, and $\mathbf{E}$ and $\mathbf{B}$ are three-vectors of the electric and magnetic fields, respectively. Thus our electromagnetic field is non-null. Metric (2) is of Petrov type D. Its two Debever–Penrose vectors k^{μ} and l^{μ}, each double of course, can have the covariant components $k_{\mu} = \delta^{u}_{\mu}$ and $l_{\mu} = \delta^{v}_{\mu}$; they are geodesic, shear-free, rotation-free, and expansion-free. These are also principal

null vectors of our electromagnetic field, i.e. we have here a doubly aligned case. From Eq. (4) we see that $\Lambda \leq 0$ in our solution.

The general form of our solution is $ds^2 = dx^2 + dy^2 + 2e^A\, du\, dv$, where a disposable function $A = A(u, v)$ is restricted by the condition $e^{-A} A_{,uv} = -2\Lambda$, but owing to this condition we can retransform u and v so as to get rid of the disposable function and obtain the metric (2). For a proof see Refs. [7,8].

After making a coordinate transformation

$$x = x, \quad y = y, \quad u = 2^{1/2} j^{-1} e^{jz} M, \quad v = -2^{1/2} j^{-1} e^{-jz} M,$$

$$j := (-2\Lambda)^{1/2}, \qquad M := \tan\left(\tfrac{1}{2} jct + \tfrac{1}{4}\pi\right), \tag{7}$$

whose Jacobian is

$$\frac{\partial(x, y, u, v)}{\partial(x, y, z, t)} = -4c\,(1 - \sin jct)^{-2} \cos jct, \tag{8}$$

we get our metric (2) in synchronous coordinates

$$ds^2 = dx^2 + dy^2 + \cos^2(jct)\; dz^2 - c^2 dt^2, \tag{9}$$

i.e. t is the cosmic time; and the Cartesian-like components of $\mathbf{E}$ and $\mathbf{B}$ are

$$E_x = E_y = 0, \qquad E_z = q = \pm|\mathbf{E}|,$$

$$B_x = B_y = 0, \qquad B_z = p = \pm|\mathbf{B}|. \tag{10}$$

Note that the limiting transition $\Lambda \to 0$ does not make the determinant (8) singular.

In all known to me exact solutions of Eqs. (5) the cosmological constant and electromagnetic field are independent. (Note added: there exists a counterpart of our solution, see the end of the present article.) Solution (2)–(4) is therefore a surprise. In virtue of Eqs. (4) and (6) the existence of fields (10) is *equivalent* to the existence of a negative Λ. The fields (10) constitute a *constant electromagnetic background* in the whole universe (if and only if $\Lambda < 0$). This background is not

uniquely determined by the value of Λ since p and q are independent. For every given $\Lambda < 0$ we can have, by Eqs. (10), arbitrary values of $\mathbf{E}^2$ and $\mathbf{B}^2$ within Eq. (4), including the extreme cases $\Lambda = -c^{-4}G\mathbf{E}^2$ for $\mathbf{B} = 0$ and $\Lambda = -c^{-4}G\mathbf{B}^2$ for $\mathbf{E} = 0$. It is seen from Eqs. (10) that $\mathbf{E}$ and $\mathbf{B}$ are parallel if $\mathbf{EB} \neq 0$ (or antiparallel; none of Eqs. (3)–(6) determines the signs of p and q). Thus $\mathbf{E}$ or $\mathbf{B}$ (both if $\mathbf{EB} \neq 0$) determines a *physically preferred* spacelike direction in the universe.

If one assumes that our solution describes a physical (cosmic) reality, then one admits a simple and almost *direct* method of measuring the negative cosmological constant, by searching for and measuring the electromagnetic background. If none of the fields (10) is discovered, then one concludes that the absolute value of $\Lambda < 0$ lies below the sensitivity threshold of the measurement, and the possibility of $\Lambda \geq 0$ is admitted. There are, however, Maxwellian plasmas in the interstellar and intergalactic spaces [5]. We have to assume therefore that $|\mathbf{E}|$ is extremely small or even zero since otherwise such plasmas could not exist. Then Eqs. (4) and (10) give

$$\Lambda \cong -c^{-4}G\mathbf{B}^2. \tag{11}$$

The proposed method of measuring the negative Λ is simple in principle though not necessarily in practice. In our cosmic neighbourhood we have various complicated structures of relatively strong magnetic fields [9–11], which can conceal the presumable background magnetic field. The background should therefore be sought in the large-scale extragalactic space. Unfortunately, in this wider scale we have an analogous situation, though the structures are, in general, considerably larger [12,13] and the fields considerably weaker [5,14]. Lemoine *et al.* [14] estimate the extragalactic magnetic field strength at $\sim$ 1 pG – $\sim$ 1 nG, however, this concerns the root-mean-squared strength (denoted by $B_{\rm rms}$ in Ref. [14]). This quantity is used from necessity because in general we do not know the directions and senses of almost constant magnetic fields occurring in large

extragalactic regions of space [12]. $B_{\rm rms}$ includes, by definition, uncertainties and therefore cannot be directly related to our background field $\mathbf{B}$. For instance, if we perform observations through the large regions just mentioned, and if the senses of magnetic fields in these regions differ, then the measured $B_{\rm rms}$ may be considerably lower than the field strength in some of the regions. Nevertheless, it seems obvious that the strength $|\mathbf{B}|$ of our background cannot be much higher than the values estimated above. Assuming tentatively $|\mathbf{B}| \lesssim 10$ nG, from relation (11) we get

$$|\Lambda| \lesssim 8.3 \times 10^{-66}\ \mathrm{cm}^{-2}, \tag{12}$$

i.e. values considerably lower than the upper limit in relation (1), and so small that one might even doubt whether $\Lambda < 0$ exists at all (cf. remark in reference 18 on p. 71 in Ref. [6]). On the other hand, we may not *a priori* exclude the existence of the background, but the relevant settlement would need laborious observations which should take into account the possibility of the background field occurring.

Our solution is of course too simple to represent a model of the universe. It can just be a "background addition" to the present-day models based on dust or fluid type solutions. For instance, if one assumes an isotropic model, then our solution can introduce an anisotropic perturbation (preferred direction); the lower the $|\Lambda|$ the smaller the deviation from isotropy (relation (11)).

Let us still note another aspect of our solution. If $\Lambda < 0$, then metric (9) degenerates for $t = T_n := \left(n + \frac{1}{2}\right)\pi/jc$ and $n = 0, \pm 1, \pm 2, \ldots$ (one space dimension vanishes). Thus each T_n may be interpreted as an instant of the death of a universe and of the birth of the next one. The solution suggests therefore the existence of an infinite sequence of universes; and then a period

$$T := T_{n+1} - T_n = \pi/jc \tag{13}$$

would be the entire lifetime (from birth to death) of each of them. Relations (12) and (13) give $T \gtrsim 0.8$ Pyr, i.e. a period almost 10^5 times longer than the recent

estimation [6] of the age of our universe!

Note added. Our solution is a special case of that found by Bertotti [15]. His solution also describes a spacetime with constant electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields that also are independent and parallel or antiparallel (if $\mathbf{EB} \neq 0$). When the coordinate system of Eq. (9) is used the Bertotti solution takes the form

$$ds^2 = dx^2 + \cos^2(ax)\ dy^2 + \cos^2(bct)\ dz^2 - c^2dt^2, \tag{14}$$

where the real constants a and b are determined by equations

$$a^2 = c^{-4}G\left(\mathbf{B}^2 + \mathbf{E}^2\right) + \Lambda$$

$$b^2 = c^{-4}G\left(\mathbf{B}^2 + \mathbf{E}^2\right) - \Lambda, \tag{15}$$

and the axis z is then parallel to the three-vectors $\mathbf{E}$ and $\mathbf{B}$. In this case, unlike our solution ($a = 0$), the energy density of the background electromagnetic field is independent of Λ. The metric form (14) is of Petrov type D iff $a^2 \neq b^2$ (i.e. iff $\Lambda \neq 0$ for the solution (14) and (15)). Its special case $a^2 = b^2$ (i.e. iff $\Lambda = 0$) is conformally flat and is called the Robinson solution [16]. Another special case, $a \neq 0$ and $b = 0$, is also an exact solution of the current-free Einstein–Maxwell equations. This solution is of Petrov type D and has properties analogous to those of our solution but with $\Lambda > 0$ and periodicity in space.

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